Available online at www.sciencedirect.com

SCIENCE DIRECT

ELSEVIER

Linear Algebra and its Applications 416 (2006) 880–889

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

The determinant of $AA^* - A^*A$ for a Leonard pair A, A^*

Kazumasa Nomura [a,*], Paul Terwilliger [b]

[a] *College of Liberal Arts and Sciences, Tokyo Medical and Dental University, Kohnodai, Ichikawa 272-0827, Japan*

[b] *Department of Mathematics, University of Wisconsin, Van Vleck Hall, 480 Lincoln drive, Madison, WI 53706-1388, USA*

Received 13 November 2005; accepted 31 December 2005

Available online 13 February 2006

Submitted by R.A. Brualdi

Abstract

Let $\mathbb{K}$ denote a field, and let V denote a vector space over $\mathbb{K}$ with finite positive dimension. We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy (i), (ii) below:

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.

(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

We call such a pair a *Leonard pair* on V. In this paper we investigate the commutator $AA^* - A^*A$. Our results are as follows. Abbreviate $d = \dim V - 1$ and first assume d is odd. We show $AA^* - A^*A$ is invertible and display several attractive formulae for the determinant. Next assume d is even. We show that the null space of $AA^* - A^*A$ has dimension 1. We display a nonzero vector in this null space. We express this vector as a sum of eigenvectors for A and as a sum of eigenvectors for A^*.

AMS classification: 15A15; 05E35; 05E30; 33C45; 33D45

Keywords: Leonard pair; Terwilliger algebra; Askey scheme; q-Racah polynomial

* Corresponding author.

E-mail addresses: nomura.las@tmd.ac.jp (K. Nomura), terwilli@math.wisc.edu (P. Terwilliger).

0024-3795/$ - see front matter © 2006 Elsevier Inc. All rights reserved.

doi:10.1016/j.laa.2005.12.029

1. Introduction

Throughout this paper, $\mathbb{K}$ will denote a field and V will denote a vector space over $\mathbb{K}$ with finite positive dimension.

We begin by recalling the notion of a Leonard pair. We will use the following notation. A square matrix X is called *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume X is tridiagonal. Then X is called *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

Definition 1.1 [18]. By a *Leonard pair* on V, we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following two conditions:

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

Note 1.2. It is a common notational convention to use A^* to represent the conjugate-transpose of A. We are *not* using this convention. In a Leonard pair A, A^* the linear transformations A and A^* are arbitrary subject to (i) and (ii) above.

We refer the reader to [3,10,13–18,20–27,29,30] for background on Leonard pairs. We especially recommend the survey [27]. See [1,2,5–9,11,12,19,28] for related topics.

For the rest of this paper let A, A^* denote a Leonard pair on V. For notational convenience we define $d = \dim V - 1$. We are going to investigate the commutator $AA^* - A^*A$. It turns out the behavior of this commutator depends on the parity of d. First assume d is odd. We show $AA^* - A^*A$ is invertible and display several attractive formulae for the determinant. Next assume d is even. We show that the null space of $AA^* - A^*A$ has dimension 1. We display a nonzero vector in this null space. We express this vector as a sum of eigenvectors for A and as a sum of eigenvectors for A^*.

The rest of this section is devoted to giving formal statements of our results. We will use the following notation. Let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of all $d+1$ by $d+1$ matrices that have entries in $\mathbb{K}$. We index the rows and columns by $0, 1, \ldots, d$. Let $u_0, u_1, \ldots, u_d$ denote a basis for V. For a linear transformation $X : V \to V$ and for $Y \in \mathrm{Mat}_{d+1}(\mathbb{K})$, we say Y *represents X with respect to $u_0, u_1, \ldots, u_d$* whenever $Xu_j = \sum_{i=0}^{d} Y_{ij}u_i$ for $0 \leqslant j \leqslant d$. Also, by the *null space* of X we mean $\{v \in V | Xv = 0\}$. We recall that X is invertible if and only if the null space of X is zero. We now state our first main result.

Theorem 1.3. *The following hold*:

(i) *Suppose d is odd. Then $AA^* - A^*A$ is invertible.*
(ii) *Suppose d is even. Then the null space of $AA^* - A^*A$ has dimension* 1.

Before we state our next two main theorems we make some comments. We fix a basis $v_0^*, v_1^*, \ldots, v_d^*$ for V that satisfies Definition 1.1(i). Observe that with respect to this basis the matrices that represent A, A^* take the form

$$A:\begin{pmatrix} a_0 & b_0 & & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{d-1} \\ \mathbf{0} & & & & c_d & a_d \end{pmatrix}, \quad A^*:\begin{pmatrix} \theta_0^* & & & & & \mathbf{0} \\ & \theta_1^* & & & & \\ & & \theta_2^* & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ \mathbf{0} & & & & & \theta_d^* \end{pmatrix}, \tag{1}$$

where $b_{i-1}c_i \neq 0$ for $1 \leqslant i \leqslant d$. We also fix a basis $v_0, v_1, \ldots, v_d$ for V that satisfies Definition 1.1(ii). With respect to this basis the matrices that represent A, A^* take the form

$$A:\begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ & \theta_1 & & & & \\ & & \theta_2 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ \mathbf{0} & & & & & \theta_d \end{pmatrix}, \quad A^*:\begin{pmatrix} a_0^* & b_0^* & & & & \mathbf{0} \\ c_1^* & a_1^* & b_1^* & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{d-1}^* \\ \mathbf{0} & & & & c_d^* & a_d^* \end{pmatrix}, \tag{2}$$

where $b_{i-1}^*c_i^* \neq 0$ for $1 \leqslant i \leqslant d$. Observe that $\theta_0, \theta_1, \ldots, \theta_d$ (respectively $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$) are the eigenvalues of A (respectively A^*). It is known [18, Lemma 1.3] that $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ if $i \neq j$ for $0 \leqslant i, j \leqslant d$. We now state our second and third main results.

Theorem 1.4. *Suppose d is odd. Then*

$$\det(AA^* - A^*A) = \prod_{\substack{1\leqslant i\leqslant d \\ i \text{ odd}}} b_{i-1}c_i\left(\theta_{i-1}^* - \theta_i^*\right)^2, \tag{3}$$

$$\det(AA^* - A^*A) = \prod_{\substack{1\leqslant i\leqslant d \\ i \text{ odd}}} b_{i-1}^*c_i^*(\theta_{i-1} - \theta_i)^2. \tag{4}$$

Theorem 1.5. *Suppose d is even.*

(i) *The null space of $AA^* - A^*A$ is spanned by $\sum_{k=0}^{d} \gamma_k v_k^*$, where $\gamma_k = 0$ if k is odd, and*

$$\gamma_k = \prod_{\substack{1\leqslant i\leqslant k-1 \\ i \text{ odd}}} \frac{c_i\left(\theta_{i-1}^* - \theta_i^*\right)}{b_i\left(\theta_i^* - \theta_{i+1}^*\right)}, \tag{5}$$

if k is even.

(ii) *The null space of $AA^* - A^*A$ is spanned by $\sum_{k=0}^{d} \gamma_k^* v_k$, where $\gamma_k^* = 0$ if k is odd, and*

$$\gamma_k^* = \prod_{\substack{1\leqslant i\leqslant k-1 \\ i \text{ odd}}} \frac{c_i^*(\theta_{i-1} - \theta_i)}{b_i^*(\theta_i - \theta_{i+1})}, \tag{6}$$

if k is even.

Remark 1.6. Theorems 1.3 and 1.5 give an answer to a problem by the second author [27, Section 36].

In order to state our next result we recall a few facts.

Lemma 1.7 [18, Theorem 1.9]. *The expressions*

$$\frac{\theta_{i-2}-\theta_{i+1}}{\theta_{i-1}-\theta_i}, \qquad \frac{\theta^*_{i-2}-\theta^*_{i+1}}{\theta^*_{i-1}-\theta^*_i} \tag{7}$$

are equal and independent of i for $2 \leqslant i \leqslant d-1$.

Using Lemma 1.7 we define a scalar q as follows.

Definition 1.8. For $d \geqslant 3$ let β denote the scalar in $\mathbb{K}$ such that $\beta+1$ is the common value of (7). For $d \leqslant 2$ let β denote any scalar in $\mathbb{K}$. Let $\overline{\mathbb{K}}$ denote the algebraic closure of $\mathbb{K}$. Let q denote a nonzero scalar in $\overline{\mathbb{K}}$ such that $\beta = q^2+q^{-2}$.

We recall some notation.

Definition 1.9. For an integer $n > 0$ we define

$$[n]_q = q^{n-1}+q^{n-3}+\cdots+q^{1-n}. \tag{8}$$

We observe:

(i) If $q^2 \neq 1$ then

$$[n]_q = \frac{q^n-q^{-n}}{q-q^{-1}}. \tag{9}$$

(ii) If $q = 1$ then

$$[n]_q = n. \tag{10}$$

(iii) If $q=-1$ then

$$[n]_q = (-1)^{n+1}n. \tag{11}$$

We mention here a technical result for later use. We will show

$$[i]_q \neq 0 \quad \text{if } i \text{ is odd} \quad (1 \leqslant i \leqslant d). \tag{12}$$

We recall some parameters. By [18, Theorem 3.2] there exists a sequence of nonzero scalars $\varphi_1, \varphi_2, \ldots, \varphi_d$ in $\mathbb{K}$ and there exists a basis for V with respect to which the matrices representing A, A^* are

$$A: \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \qquad A^*: \begin{pmatrix} \theta^*_0 & \varphi_1 & & & & \mathbf{0} \\ & \theta^*_1 & \varphi_2 & & & \\ & & \theta^*_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta^*_d \end{pmatrix}.$$

The sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$ is uniquely determined by the ordering $(\theta_0, \theta_1, \ldots, \theta_d; \theta^*_0, \theta^*_1, \ldots, \theta^*_d)$. We call the sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$ the *first split sequence* with respect to the ordering $(\theta_0, \theta_1, \ldots, \theta_d; \theta^*_0, \theta^*_1, \ldots, \theta^*_d)$. Let $\phi_1, \phi_2, \ldots, \phi_d$ denote the first split sequence with respect to the ordering $(\theta_d, \theta_{d-1}, \ldots, \theta_0; \theta^*_0, \theta^*_1, \ldots, \theta^*_d)$. We call the sequence $\phi_1, \phi_2, \ldots, \phi_d$ the *second split sequence* with respect to the ordering $(\theta_0, \theta_1, \ldots, \theta_d; \theta^*_0, \theta^*_1, \ldots, \theta^*_d)$. We now state our final main result.

Theorem 1.10. *Suppose d is odd. Then*

$$\det(AA^* - A^*A) = (-1)^{(d+1)/2} \prod_{\substack{1 \leqslant i \leqslant d \\ i \text{ odd}}} \frac{\varphi_i \phi_i}{[i]_q^2}. \tag{13}$$

Remark 1.11. The denominator of (13) is nonzero by (12).

Remark 1.12. Theorem 1.10 was conjectured by the second author [27, Section 36].

2. Proof of Theorems 1.3, 1.4 and 1.5

Lemma 2.1. *Let $B \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the matrix that represents $AA^* - A^*A$ with respect to the basis $v^*_0, v^*_1, \ldots, v^*_d$. Then*

(i) *The $(i, i-1)$-entry of B is $c_i\left(\theta^*_{i-1} - \theta^*_i\right)$ for $1 \leqslant i \leqslant d$.*
(ii) *The $(i-1, i)$-entry of B is $b_{i-1}\left(\theta^*_i - \theta^*_{i-1}\right)$ for $1 \leqslant i \leqslant d$.*
(iii) *All other entries of B are* 0.

Proof. Obtained by routine computation using (1). □

Proof of Theorem 1.4. We first prove (3). Let the matrix B be as in Lemma 2.1. Observe that B is tridiagonal with all diagonal entries zero. For $0 \leqslant r \leqslant d$ let B_r denote the submatrix of B obtained by taking rows $0, 1, \ldots, r$ and columns $0, 1, \ldots, r$. Then the determinants of $B_1, B_3, \ldots, B_d$ satisfy the following well-known recursion [4, p. 28]:

$$\det(B_1) = b_0 c_1 \left(\theta^*_0 - \theta^*_1\right)^2,$$
$$\det(B_r) = b_{r-1} c_r \left(\theta^*_{r-1} - \theta^*_r\right)^2 \det(B_{r-2}) \quad (3 \leqslant r \leqslant d, r \text{ odd}).$$

Solving this recursion we find

$$\det(B_r) = \prod_{\substack{1 \leqslant i \leqslant r \\ i \text{ odd}}} b_{i-1} c_i \left(\theta^*_{i-1} - \theta^*_i\right)^2 \quad (1 \leqslant r \leqslant d,\ r \text{ odd}). \tag{14}$$

Setting $r = d$ in (14) we obtain (3). The proof of (4) is similar. □

Proof of Theorem 1.5(i). Let the matrix B be as in Lemma 2.1. Define a vector $v = (\gamma_0, \gamma_1, \ldots, \gamma_d)^{\mathrm{t}}$ where t denotes the transpose, and $\gamma_0, \gamma_1, \ldots, \gamma_d$ are from the statement of the theorem. It suffices to show that v spans the null space of B. By matrix multiplication we find $Bv = 0$, so v is contained in the null space of B. Let w denote any vector in the null space of B. We show w is a scalar multiple of v. For notational convenience write $w = (w_0, w_1, \ldots, w_d)^{\mathrm{t}}$. Multiplying out $Bw = 0$ we routinely obtain the recursion

$$b_0\left(\theta^*_1 - \theta^*_0\right) w_1 = 0,$$

$$c_i\left(\theta^*_{i-1}-\theta^*_i\right)w_{i-1}+b_i\left(\theta^*_{i+1}-\theta^*_i\right)w_{i+1}=0 \quad (1 \leqslant i \leqslant d-1),$$
$$c_d\left(\theta^*_{d-1}-\theta^*_d\right)w_{d-1}=0.$$

Solving this recursion we find $w_k = w_0\gamma_k$ for $0 \leqslant k \leqslant d$. Therefore $w = w_0 v$. We have now shown that w is a scalar multiple of v and the result follows. □

Proof of Theorem 1.5(ii). Similar to the proof of Theorem 1.5(i). □

Proof of Theorem 1.3. Immediate from Theorems 1.4 and 1.5. □

3. The proof of Theorem 1.10, part I

We now turn to the proof of Theorem 1.10. We will use the following notation. Let λ denote an indeterminate and let $\mathbb{K}[\lambda]$ denote the $\mathbb{K}$-algebra consisting of all polynomials in λ that have coefficients in $\mathbb{K}$.

Definition 3.1. For $0 \leqslant i \leqslant d$ let τ^*_i, η^*_i denote the following polynomials in $\mathbb{K}[\lambda]$:

$$\tau^*_i=\left(\lambda-\theta^*_0\right)\left(\lambda-\theta^*_1\right)\cdots\left(\lambda-\theta^*_{i-1}\right), \tag{15}$$
$$\eta^*_i=\left(\lambda-\theta^*_d\right)\left(\lambda-\theta^*_{d-1}\right)\cdots\left(\lambda-\theta^*_{d-i+1}\right). \tag{16}$$

We observe that each of τ^*_i, η^*_i is monic with degree i.

Theorem 3.2 [27, Lemma 7.2, Theorem 23.7]. *For $1 \leqslant i \leqslant d$ we have*

$$b_{i-1}c_i = \varphi_i\phi_i\,\frac{\tau^*_{i-1}\left(\theta^*_{i-1}\right)\eta^*_{d-i}\left(\theta^*_i\right)}{\tau^*_i\left(\theta^*_i\right)\eta^*_{d-i+1}\left(\theta^*_{i-1}\right)}. \tag{17}$$

We now assume that d is odd and evaluate (3) using (17). We find that $\det(AA^*-A^*A)$ is equal to

$$\prod_{\substack{1\leqslant i\leqslant d\\ i\ \text{odd}}}\varphi_i\phi_i$$

times

$$\prod_{\substack{1\leqslant i\leqslant d\\ i\ \text{odd}}}\left(\theta^*_{i-1}-\theta^*_i\right)^2\frac{\tau^*_{i-1}\left(\theta^*_{i-1}\right)\eta^*_{d-i}\left(\theta^*_i\right)}{\tau^*_i\left(\theta^*_i\right)\eta^*_{d-i+1}\left(\theta^*_{i-1}\right)}. \tag{18}$$

We now evaluate (18).

Lemma 3.3. *Suppose d is odd. Then (18) is equal to*

$$(-1)^{m+1}\Psi^2, \tag{19}$$

where $m=(d-1)/2$ and

$$\Psi=\prod_{0\leqslant\ell<k\leqslant m}\frac{\theta^*_{2\ell+1}-\theta^*_{2k}}{\theta^*_{2\ell}-\theta^*_{2k+1}}.$$

Proof. For an integer i define $s(i) = (-1)^i$. Using (16) we find

$$\prod_{\substack{1\leqslant i\leqslant d\\ i \text{ odd}}} \frac{\eta^*_{d-i}(\theta^*_i)}{\eta^*_{d-i+1}(\theta^*_{i-1})} = \prod_{0\leqslant i<j\leqslant d} (\theta^*_i - \theta^*_j)^{s(i+1)}.$$

Similarly using (15) we find

$$\prod_{\substack{1\leqslant i\leqslant d\\ i \text{ odd}}} \frac{\tau^*_{i-1}(\theta^*_{i-1})}{\tau^*_i(\theta^*_i)} = (-1)^{m+1} \prod_{0\leqslant i<j\leqslant d} (\theta^*_i - \theta^*_j)^{s(j)}.$$

Evaluating (18) using these equations we routinely obtain the result. □

4. Some comments

In order to prove Theorem 1.10 we will evaluate (19) further using Lemma 1.7. There are some technical aspects involved which we will deal with in this section.

Lemma 4.1 [18, Lemma 9.3]. *Assume $d \geqslant 3$. Then with reference to Definition 1.8 the following* (i)–(iv) *hold*:

(i) *Suppose $\beta \neq 2, \beta \neq -2$. Then $q^{2i} \neq 1$ for $1 \leqslant i \leqslant d$.*
(ii) *Suppose $\beta = 2$ and $\mathrm{Char}(\mathbb{K}) = p > 2$. Then $d < p$.*
(iii) *Suppose $\beta = -2$ and $\mathrm{Char}(\mathbb{K}) = p > 2$. Then $d < 2p$.*
(iv) *Suppose $\beta = 0$ and $\mathrm{Char}(\mathbb{K}) = 2$. Then $d = 3$.*

Lemma 4.2. *Referring to Definition 1.9, assume that n is odd and $q^2 = -1$. Then $[n]_q = (-1)^{(n-1)/2}$.*

Proof. Routine using line (9). □

Corollary 4.3. *With reference to Definitions 1.8 and 1.9, we have $[i]_q \neq 0$ for i odd $(1 \leqslant i \leqslant d)$.*

Proof. Assume $d \geqslant 3$; otherwise the result holds since $[1]_q = 1$. Let the integer i be given and assume i is odd. We consider three cases. First assume $\beta \neq 2, \beta \neq -2$. Then the result holds by Lemma 4.1(i) and (9). Next assume $\beta = 2$ and $\mathrm{Char}(\mathbb{K}) \neq 2$. Using $\beta = 2$ and $q^2 + q^{-2} = \beta$ we find $q^2 = 1$. Now $[i]_q = i$ by (10) or (11) and since i is odd. Each of $1, 2, \ldots, d$ is nonzero in $\mathbb{K}$ by Lemma 4.1(ii) so $[i]_q \neq 0$. Next assume $\beta = -2$. Using $q^2 + q^{-2} = \beta$ we find $q^2 = -1$, so $[i]_q = (-1)^{(i-1)/2}$ by Lemma 4.2. In particular $[i]_q \neq 0$ as desired. □

Lemma 4.4 [18, Lemma 9.4]. *Assume $d \geqslant 3$. Pick any integers i, j, r, s $(0 \leqslant i, j, r, s \leqslant d)$ and assume $i + j = r + s, i \neq j, r \neq s$. Then with reference to Definition 1.8 the following* (i)–(iv) *hold*:

(i) *Suppose $\beta \neq 2, \beta \neq -2$. Then*

$$\frac{\theta^*_i - \theta^*_j}{\theta^*_r - \theta^*_s} = \frac{q^{2i} - q^{2j}}{q^{2r} - q^{2s}}.$$

(ii) *Suppose* $\beta = 2$ *and* $\mathrm{Char}(\mathbb{K}) \neq 2$. *Then*

$$\frac{\theta^*_i - \theta^*_j}{\theta^*_r - \theta^*_s} = \frac{i-j}{r-s}.$$

(iii) *Suppose* $\beta = -2$ *and* $\mathrm{Char}(\mathbb{K}) \neq 2$. *If* $r+s$ *is even, then*

$$\frac{\theta^*_i - \theta^*_j}{\theta^*_r - \theta^*_s} = (-1)^{i+r}\frac{i-j}{r-s}.$$

If $r+s$ *is odd, then*

$$\frac{\theta^*_i - \theta^*_j}{\theta^*_r - \theta^*_s} = (-1)^{i+r}.$$

(iv) *Suppose* $\beta = 0$ *and* $\mathrm{Char}(\mathbb{K}) = 2$. *Then*

$$\frac{\theta^*_i - \theta^*_j}{\theta^*_r - \theta^*_s} = 1.$$

In the above formulae all denominators are nonzero by Lemma 4.1.

Corollary 4.5. *Assume* $d \geqslant 3$. *Pick any integers* i, j $(0 \leqslant i < j \leqslant d)$ *and* r, s $(0 \leqslant r < s \leqslant d)$. *Assume* $i + j = r + s$ *and this common value is odd. Then with reference to Definitions* 1.8 *and* 1.9,

$$\frac{\theta^*_i - \theta^*_j}{\theta^*_r - \theta^*_s} = \frac{[j-i]_q}{[s-r]_q}. \tag{20}$$

Proof. In each case of Lemma 4.4 we routinely express the result using Definition 1.9 and Lemma 4.2. □

5. Proof of Theorem 1.10, part II

In this section we complete the proof of Theorem 1.10. Our argument is based on the following proposition.

Proposition 5.1. *Assume* d *is odd. Then the expression* Ψ *from Lemma* 3.3 *satisfies*

$$\Psi = \prod_{\substack{1 \leqslant i \leqslant d \\ i \text{ odd}}} \frac{1}{[i]_q}. \tag{21}$$

Proof. We may assume $d \geqslant 3$; otherwise the result holds since $[1]_q = 1$. Now we have

$$\begin{aligned}\Psi &= \prod_{0 \leqslant \ell < k \leqslant m} \frac{\theta^*_{2\ell+1} - \theta^*_{2k}}{\theta^*_{2\ell} - \theta^*_{2k+1}} \\ &= \prod_{k=0}^{m} \prod_{\ell=0}^{k-1} \frac{\theta^*_{2\ell+1} - \theta^*_{2k}}{\theta^*_{2\ell} - \theta^*_{2k+1}}\end{aligned}$$

$$= \prod_{k=0}^{m} \prod_{\ell=0}^{k-1} \frac{[2k-2\ell-1]_q}{[2k-2\ell+1]_q} \quad \text{(by Corollary 4.5)}$$

$$= \prod_{k=0}^{m} \frac{1}{[2k+1]_q}$$

$$= \prod_{\substack{1 \leqslant i \leqslant d \\ i \text{ odd}}} \frac{1}{[i]_q}. \qquad \square$$

Proof of Theorem 1.10. Immediate from Lemma 3.3, Proposition 5.1, and the comment after Theorem 3.2. $\square$

References

[1] H. Alnajjar, B. Curtin, A family of tridiagonal pairs, Linear Algebra Appl. 390 (2004) 369–384.
[2] H. Alnajjar, B. Curtin, A family of tridiagonal pairs related to the quantum affine algebra $U_q(\widehat{sl}_2)$, Electron. J. Linear Algebra 13 (2005) 1–9.
[3] B. Hartwig, Three mutually adjacent Leonard pairs, Linear Algebra Appl. 408C (2005) 19–39.
[4] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1990.
[5] T. Ito, K. Tanabe, P. Terwilliger, Some algebra related to P- and Q-polynomial association schemes, in: Codes and Association Schemes, Piscataway NJ, 1999, Amer. Math. Soc., Providence, RI, 2001, pp. 167–192. arXiv:math.CO/0406556.
[6] T. Ito, P. Terwilliger, The shape of a tridiagonal pair, J. Pure Appl. Algebra 188 (2004) 145–160. arXiv: math.QA/0304244.
[7] T. Ito, P. Terwilliger, Tridiagonal pairs and the quantum affine algebra $U_q(\widehat{sl}_2)$, Ramanujan J., in press. arXiv:math.QA/0310042.
[8] T. Ito, P. Terwilliger, Two non-nilpotent linear transformations that satisfy the cubic q-Serre relations, J. Algebra, submitted for publication. arXiv:math.QA/0508398.
[9] T. Ito, P. Terwilliger, C. Weng, The quantum algebra $U_q(sl_2)$ and its equitable presentation, J. Algebra, in press. arXiv:math.math.QA/0507477.
[10] K. Nomura, Tridiagonal pairs and the Askey–Wilson relations, Linear Algebra Appl. 397 (2005) 99–106.
[11] K. Nomura, A refinement of the split decomposition of a tridiagonal pair, Linear Algebra Appl. 403 (2005) 1–23.
[12] K. Nomura, Tridiagonal pairs of height one, Linear Algebra Appl. 403 (2005) 118–142.
[13] K. Nomura, P. Terwilliger, Balanced Leonard pairs, Linear Algebra Appl., submitted for publication. arXiv: math.RA/0506219.
[14] K. Nomura, P. Terwilliger, Some formulae involving the split sequences of a Leonard pair, Linear Algebra Appl. 413 (2006) 189–201. arXiv:math.RA/0508407.
[15] A.A. Pascasio, On the multiplicities of the primitive idempotents of a Q-polynomial distance-regular graph, European J. Combin. 23 (2002) 1073–1078.
[16] P. Terwilliger, The subconstituent algebra of an association scheme I, J. Algebraic Combin. 1 (1992) 363–388.
[17] P. Terwilliger, The subconstituent algebra of an association scheme III, J. Algebraic Combin. 2 (1993) 177–210.
[18] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other, Linear Algebra Appl. 330 (2001) 149–203. arXiv:math.RA/0406555.
[19] P. Terwilliger, Two relations that generalize the q-Serre relations and the Dolan–Grady relations, Physics and Combinatorics 1999, Nagoya, World Scientific Publishing, River Edge, NJ, 2001, pp. 377–398. arXiv:math.QA/0307016.
[20] P. Terwilliger, Leonard pairs from 24 points of view, Rocky Mountain J. Math. 32 (2) (2002) 827–888. arXiv:math.RA/0406577.
[21] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other; the *TD–D* and the *LB–UB* canonical form, J. Algebra 291 (2005) 1–45. arXiv:math.RA/0304077.
[22] P. Terwilliger, Introduction to Leonard pairs, OPSFA Rome 2001, J. Comput. Appl. Math. 153 (2) (2003) 463–475.

[23] P. Terwilliger, Introduction to Leonard pairs and Leonard systems, Sūrikaisekikenkyūsho Kōkyūroku 1109 (1999) 67–79, Algebraic combinatorics (Kyoto, 1999).
[24] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other; comments on the split decomposition, J. Comput. Appl. Math. 178 (2005) 437–452. arXiv:math.RA/0306290.
[25] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other; comments on the parameter array, Des. Codes Cryptogr. 34 (2005) 307–332. arXiv:math.RA/0306291.
[26] P. Terwilliger, Leonard pairs and the q-Racah polynomials, Linear Algebra Appl. 387 (2004) 235–276. arXiv:math.QA/0306301.
[27] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other; an algebraic approach to the Askey scheme of orthogonal polynomials, Lecture notes for the summer school on orthogonal polynomials and special functions, Universidad Carlos III de Madrid, Leganes, Spain, July 8–July 18, 2004. arXiv:math.QA/0408390.
[28] P. Terwilliger, The equitable presentation for the quantum group $U_q(g)$ associated with a symmetrizable Kac-Moody algebra g, J. Algebra, in press. arXiv:math.QA/0507478.
[29] P. Terwilliger, R. Vidunas, Leonard pairs and the Askey–Wilson relations, J. Algebra Appl. 3 (2004) 411–426. arXiv:math.QA/0305356.
[30] R. Vidunas, Normalized Leonard pairs and Askey–Wilson relations, Preprint. arXiv:math.RA/0505041.